International Journal of Theoretical Physics, Vol. 41, No. 8, August 2002 (© 2002)

Probability Collectives for Unstable Particles

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Received May 15, 2002

Unstable particles, together with their stable decay products, constitute probability collectives that are defined as Hilbert spaces with dimension higher than one, nondecomposable in a particle basis. Their structure is considered in the framework of Birkhoff-von Neumann's Hilbert subspace lattices. Bases with particle states are related to bases with a diagonal scalar product by a Hilbert-bein involving the characteristic decay parameters (in some analogy to the *n*-bein structures of metrical manifolds). Probability predictions as expectation values, involving unstable particles, have to take into account all members of the higher dimensional collective. For example, the unitarity structure of the *S*-matrix for an unstable particle collective can be established by a transformation with its Hilbert-bein.

KEY WORDS: probability collectives; unstable particles.

1. STABLE PARTICLE HILBERT SPACES

The Hilbert space used for stable particles with mass m, momentum $\vec{p}$, and possible homogeneous degrees of freedom $a = 1, 2, \ldots, K$—including particles and antiparticles with spin and internal degrees of freedom—comes with creation operators $\mathrm{u}^a(m, \vec{p})$ and annihilation operators $u^*_a(m, \vec{p})$. To have the involved concepts and notations at hand, it is shortly reviewed by repeating its construction.

The underlying quantum structure for Bose (commutator $\epsilon = -1$) and Fermi (anticommutator $\epsilon = +1$)

$$[\mathrm{u}^*, \mathrm{u}]_\epsilon = 1, \qquad [\mathrm{u}, \mathrm{u}]_\epsilon = 0 = [\mathrm{u}^*, \mathrm{u}^*]_\epsilon$$

comes with the time translation behavior of the Bose and Fermi harmonic oscillator as implemented by the Hamiltonian

$$H = E\frac{[\mathrm{u}, \mathrm{u}^*]_{-\epsilon}}{2}$$

involving the quantum-opposite commutator and a frequency (energy) scale $E > 0$.

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0020-7748/02/0800-1491/0 © 2002 Plenum Publishing Corporation

Creation and annihilation operators build by the linear combinations of their products the quantum algebras $\mathbf{Q}_\epsilon(\mathbb{C}^2)$, countably infinite dimensional for Bose and 4-dimensional for Fermi

$$\mathbf{Q}_\epsilon(\mathbb{C}^2) = \mathbb{C}[\mathrm{u}, \mathrm{u}^*]/\text{modulo} \begin{Bmatrix} [\mathrm{u}^*, \mathrm{u}]_\epsilon - 1 \\ [\mathrm{u}, \mathrm{u}]_\epsilon, \quad [\mathrm{u}^*, \mathrm{u}^*]_\epsilon \end{Bmatrix}$$

$$\cong \begin{cases} \mathbb{C}^{\aleph_0} & \text{Bose}, \quad \epsilon = -1 \\ \mathbb{C}^4 & \text{Fermi}, \quad \epsilon = +1 \end{cases}$$

$$\text{basis of } \mathbf{Q}_\epsilon(\mathbb{C}^2) : \begin{cases} \{\mathrm{u}^k(\mathrm{u}^*)^l \mid k, l = 0, 1, \ldots\} & \text{Bose} \\ \{1, \mathrm{u}, \mathrm{u}^*, \mathrm{u}\mathrm{u}^*\} & \text{Fermi} \end{cases}$$

The given bases contain eigenvectors of the time translations, $\frac{d}{dt}a = [iH, a]$

$$[H, \mathrm{u}^k](\mathrm{u}^*)^l] = (k - l)E\mathrm{u}^k(\mathrm{u}^*)^l$$

The Fock state $\langle\ \rangle_\mathrm{F}$, a conjugation compatible linear form of the quantum algebras, is induced by the scalar product $\langle \mathrm{u} \mid \mathrm{u}\rangle = 1$ of the 1-dimensional basic vector space $\mathbb{C}\mathrm{u}$. The scalar product invariance group $\mathbf{U}(1)$ contains the irreducible time translation representation $t \longmapsto e^{itE} \in \mathbf{U}(1)$, generated by the Hamiltonian H

$$\mathbf{Q}_\epsilon(\mathbb{C}^2) \ni a \mapsto \langle a\rangle_\mathrm{F} \in \mathbb{C} \quad \begin{cases} \langle(\mathrm{u}^*\mathrm{u})^k\rangle_\mathrm{F} = (\langle \mathrm{u}^*\mathrm{u}\rangle_\mathrm{F})^k = 1 \;\; k = 0, 1, \ldots \\ \langle(\mathrm{u}^*)^l\mathrm{u}^k\rangle_\mathrm{F} = 0 \;\; \text{for } k \neq l \end{cases}$$

The Fock space $\mathrm{Fock}_\epsilon(\mathbb{C}^2)$ is a quotient space of the quantum algebra, constituted by the classes with respect to the elements with trivial scalar product (the annihilation left ideal in the quantum algebra)

$$\{a \in \mathbf{Q}_\epsilon(\mathbb{C}^2) \mid \langle aa^*\rangle_\mathrm{F} = 0\} = \mathbf{Q}_\epsilon(\mathbb{C}^2)\mathrm{u}^*, \qquad \mathrm{Fock}_\epsilon(\mathbb{C}^2) = \mathbf{Q}_\epsilon(\mathbb{C}^2)/\mathbf{Q}_\epsilon(\mathbb{C}^2)\mathrm{u}^*$$

$\mathrm{Fock}_\epsilon(\mathbb{C}^2)$ has a definite scalar product. The classes are called state vectors $|a\rangle$. The class $|0\rangle$ (zero quantum state vector) of the algebra unit 1 is the harmonic oscillator ground state. It is a cyclic vector for the quantum algebra action with the annihilation property $\mathrm{u}^*|0\rangle = 0$. The state vectors $|k\rangle$ with k quanta constitute a Fock space basis, they are time translation eigenvectors

$$\text{ground state } |0\rangle = 1 + \mathbf{Q}_\epsilon(\mathbb{C}^2)\mathrm{u}^*, \quad |k\rangle = \frac{\mathrm{u}^k}{\sqrt{k!}}|0\rangle, \quad H|K\rangle = \left(k - \frac{\epsilon}{2}\right)E|k\rangle$$

$$\text{basis of } \mathrm{Fock}_\epsilon(\mathbb{C}^2)\text{:} \begin{Bmatrix} \{|k\rangle \mid k = 0, 1, \ldots\} & \text{Bose} \\ \{|0\rangle, |1\rangle\} & \text{Fermi} \end{Bmatrix}, \ \langle k \mid l\rangle = \delta_{kl}$$

The Bose structure in quantum mechanics—not the Fermi structure—allows a position–momentum interpretation

$$\text{Bose only: } \mathbf{x} = \frac{\mathrm{u}^* + \mathrm{u}}{\sqrt{2}}, \qquad i\mathbf{p} = \frac{\mathrm{u}^* - \mathrm{u}}{\sqrt{2}} \Rightarrow \begin{cases} [\mathrm{u}^*, \mathrm{u}] = 1 = [i\mathbf{p}, \mathbf{x}] \\ H = E\,\frac{\{\mathrm{u}, \mathrm{u}^*\}}{2} = E\,\frac{\mathbf{p}^2 + \mathrm{x}^2}{2} \end{cases}$$

The familiar Schrödinger wave functions $|k\rangle \cong \psi_k(x)$ are orbits of the position translation $\mathbb{R} = \operatorname{spec}\mathbf{x} \ni x \longmapsto \psi(x) \in \mathbb{C}$.

The Fermi Fock space is a Hilbert space as well as the completion of the Bose Fock space. The Fock vector spaces are the direct sum of the totally symmetric and antisymmetric tensor powers for Bose and Fermi respectively of the complex 1-dimensional Hilbert space $\mathbb{C}|1\rangle$ with the 1-quantum state vectors of energy E denoted by $|1\rangle = |E\rangle$

$$\begin{matrix} \text{Fock}_-(\mathbb{C}^2) = \bigvee W_-(E) \cong \mathbb{C}^{\aleph_0} \\ \text{Fock}_+(\mathbb{C}^2) = \bigwedge W_+(E) \cong \mathbb{C}^2 \end{matrix} \quad \text{with} \begin{cases} W_\epsilon(E) = \ \mathbb{C}|E\rangle \cong \mathbb{C} \\ \quad |E\rangle = \ \mathrm{u}(E)\,|\,0\rangle,\ \langle E|E\rangle = 1 \end{cases}$$

Such Fock spaces will be used for different energies (frequencies) $E = p_0 = \sqrt{m^2 + p^{-2}}$.

For a stable particle with mass m, momentum $\vec{p}$ and homogeneous degrees of freedom $a = 1, \ldots, K$ one works with a direct sum-integral of Hilbert spaces, integrating with a Lorentz invariant measure the 1-quanta Hilbert spaces for all momenta

$$\bigoplus_{a=1}^{K} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3 p_0} W_\epsilon^a(m, \vec{p}) : [\mathrm{u}_b^*(m, \vec{q}), \mathrm{u}^a(m, \vec{p})]_\epsilon = \delta_b^a (2\pi)^3 p_0 \delta(\vec{p} - \vec{q})$$

with

$$\begin{aligned} p_0 &= \sqrt{m^2 + p^{-2}} \\ W_\epsilon^a(m, \vec{p}) &= \mathbb{C}|m, \vec{p}, a\rangle, \qquad |m, \vec{p}, a\rangle = \mathrm{u}^a(m, \vec{p})|0\rangle \\ \langle m, \vec{q}, b|m, \vec{p}, a\rangle &= \delta_b^a (2\pi)^3 p_0 \delta(\vec{p} - \vec{q}) \end{aligned}$$

Up to the overcountably infinite dimensional momentum dependence $\mathbb{C}^{\mathrm{R}^3}$ the 1-quantum basic Hilbert spaces used are a direct sum of a Bose and a Fermi space with—for stable particles—orthogonal subspaces for different masses

$$\begin{aligned} &W = W_+ \oplus W_- : W_\epsilon = \bigoplus_{A=1}^{s} W_\epsilon(m_A), \qquad W_\epsilon(m) = \bigoplus_{a=1}^{K} W_\epsilon^a(m) \cong \mathbb{C}^K \\ &\langle m_B, b|m_A, a\rangle = \delta_{AB}\delta_b^a \end{aligned}$$

The corresponding multiquanta states—generalizing the state vectors $|k\rangle \in \text{Fock}_\epsilon(\mathbb{C}^2)$ above—are appropriately defined tensor products.

2. UNSTABLE STATES AND PARTICLES (PART 1)

To introduce into the later, more abstract sections, the kaon system with the short and long lived unstable neutral kaon is given as an illustration.

2.1. The Collective of Neutral Kaons

The system of the two neutral K-mesons $K^0_{\mathrm{S,L}}$ (short and long) with the mass denoted state vectors $|M_{\mathrm{S,L}}\rangle$

$$M = M^0 + i\frac{\Gamma}{2},\ \ \Gamma > 0$$

spans a 2-dimensional Hilbert space. The kaon particles are no CP-eigenstates $|K_\pm\rangle$ to which they can be transformed by an invertible (2×2) matrix, called the Hilbert-bein of the neutral kaon-system

$$\begin{pmatrix}|M_{\mathrm{S}}\rangle\\|M_{\mathrm{L}}\rangle\end{pmatrix} = \xi_2\begin{pmatrix}|K_+\rangle\\|K_-\rangle\end{pmatrix}, \quad \xi_2 \in \mathbf{GL}(\mathbb{C}^2)$$

Under the assumption of CPT-invariance the matrix is symmetric and parametrizable by two complex numbers including a normalization factor N

$$\xi_2 = \frac{1}{N\sqrt{1+|\epsilon|^2}}\begin{pmatrix}1 & \epsilon\\ \epsilon & 1\end{pmatrix}, \quad \epsilon, N \in \mathbb{C}$$

There are no observable particles connected with the CP-eigenstates.

The time development is implemented by a Hamiltonian, non-hermitian for unstable particles $H_2 \neq H_2^*$

$$\text{for } t\geq 0: \frac{d}{dt}\begin{pmatrix}|K_+\rangle\\|K_-\rangle\end{pmatrix} = iH_2\begin{pmatrix}|K_+\rangle\\|K_-\rangle\end{pmatrix}, \qquad \frac{d}{dt}\begin{pmatrix}|M_{\mathrm{S}}\rangle\\|M_{\mathrm{L}}\rangle\end{pmatrix} = i\,\mathrm{diag}\,H_2\begin{pmatrix}|M_{\mathrm{S}}\rangle\\|M_{\mathrm{L}}\rangle\end{pmatrix}$$

with the diagonal form for the energy eigenstates

$$\xi_2 H_2 \xi_2^{-1} = \mathrm{diag}\,H_2 = \begin{pmatrix}M_{\mathrm{S}} & 0\\ 0 & M_{\mathrm{L}}\end{pmatrix}$$

The CP-eigenstates constitute an orthonormal basis

$$\text{CP-eigenstates: } \begin{pmatrix}\langle K_+ \mid K_+\rangle & \langle K_+ \mid K_-\rangle\\ \langle K_- \mid K_+\rangle & \langle K_- \mid K_-\rangle\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

whereas the scalar product of the energy eigenstates is given by the absolute square of the Hilbert-bein

$$\text{particles: } \zeta_2 = \xi_2\xi_2^* = \begin{pmatrix}\langle M_{\mathrm{S}} \mid M_{\mathrm{S}}\rangle & \langle M_{\mathrm{S}} \mid M_{\mathrm{L}}\rangle\\ \langle M_{\mathrm{L}} \mid M_{\mathrm{S}}\rangle & \langle M_{\mathrm{L}} \mid M_{\mathrm{L}}\rangle\end{pmatrix} = \frac{1}{|N|^2}\begin{pmatrix}1 & \delta\\ \delta & 1\end{pmatrix}$$

with

$$\delta = \frac{\epsilon + \epsilon}{1+|\epsilon|^2}, \quad 0 \leq |\delta| \leq 1$$

The experiments give a nontrivial transition between the short and long lived kaon proportional to the real part of ϵ. Therefore ζ_2 is not diagonal and ζ_2 not unitary

$$\delta \sim 0.327 \times 10^{-2} \Rightarrow \xi_2 \notin \mathbf{U}(2)$$

A decomposition of the unit operator in the 2-dimensional Hilbert space can be written with orthonormal bases, e.g. with CP-eigenstates

$$\mathbf{1}_2 = |K_+\rangle\langle K_+| + |K_-\rangle\langle K_-|$$

or with the non-orthogonal particle basis which displays the inverse scalar product matrix

$$\zeta_2^{-1} = \frac{|N|^2}{1-\delta^2}\begin{pmatrix} 1 & -\delta \\ -\delta & 1 \end{pmatrix}$$
$$\Rightarrow \mathbf{1}_2 = \frac{|N|^2}{1-\delta^2}[|M_\mathrm{S}\rangle\langle M_\mathrm{S}| - \delta|M_\mathrm{S}\rangle\langle M_\mathrm{L}| - \delta|M_\mathrm{L}\rangle\langle M_\mathrm{S}| + |M_\mathrm{L}\rangle\langle M_\mathrm{L}|]$$

2.2. Decay Collectives

The two translation eigenstates (particles) for unstable kaons $|M_{\mathrm{S,L}}\rangle$ come together with their decay products, e.g. $|\pi, \pi\rangle$, $|\pi, \pi, \pi\rangle$, $|\pi, l, \nu_l\rangle$, approximated as stable in the following. All those particles together constitute an example for a decay collective, consisting of unstable decaying particles and their stable decay products.

In general, d unstable states (particles) $\{|M_\kappa\rangle | \kappa = 1, \ldots, d\}$ spanning the space $|M\rangle \cong \mathbb{C}^d$ with complex masses $M = M^0 + i\frac{\Gamma}{2}, \Gamma > 0$, are considered together with their stable decay modes, given by s states (particles) $\{|m_a\rangle | a = 1, \ldots, s\}$ with real masses m which span the space $|m\rangle \cong \mathbb{C}^s$. All those particles are assumed to span a Hilbert space W with dimension $n = d + s$. Therein, the subspace $|m\rangle$ has an orthonormal particle basis

$$\langle m \mid m\rangle = \mathbf{1}_s$$

There are orthonormal bases $|U\rangle$ for the d-dimensional complementary space $|m\rangle^\perp \cong \mathbb{C}^d$

$$\begin{pmatrix} \langle U \mid U\rangle & \langle U \mid m\rangle \\ \langle m \mid U\rangle & \langle m \mid m\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & \mathbf{1}_s \end{pmatrix}$$

The time development in the orthonormal basis $\{|U\rangle, |m\rangle\}$ has the typical triangular form with the diagonal time development for the stable particles $m = m^* = \operatorname{diag} m$ and a nondiagonal $(d \times s)$ matrix D parametrizing the decay structure

$$\text{for } t \geq 0: \frac{d}{dt}\begin{pmatrix}|U\rangle\\|m\rangle\end{pmatrix} = iH_W\begin{pmatrix}|U\rangle\\|m\rangle\end{pmatrix}, \quad H_W = \begin{pmatrix}H_d & D\\0 & m\end{pmatrix}, \quad \frac{d}{dt}|m\rangle = im|m\rangle$$

The Hamiltonian cannot be hermitian, i.e. $H_W \neq H_W^*$—otherwise all eigenvalues would be real. It cannot even be normal in the orthonormal basis, i.e. $H_W H_W^* \neq H_W^* H_W$—otherwise it could be unitarily diagonalized diag $H_W = \xi H_W \xi^{-1}$ with $\xi \in \mathbf{U}(n)$ and, therewith, the energy eigenstates were necessarily orthogonal, having the scalar product matrix $\xi\xi^* = \mathbf{1}_n$. The Hamiltonian H_W has to be diagonalizable, i.e. its minimal polynomial has to have only order one zeros. The nonunitary diagonalization matrix

$$\xi_W H_W \xi_W^{-1} = \text{diag } H_W = \begin{pmatrix}M & 0\\0 & m\end{pmatrix}, \quad \xi_W \notin \mathbf{U}(n), \ M \neq M^*$$

called the Hilbert-bein of the decay collective, is the product of a $(d \times d)$ matrix ξ_d diagonalizing H_d—as exemplified in the kaon system of the former subsection—and a triangular matrix

$$\xi_d H_d \xi_d^{-1} = M, \qquad \xi_W = \begin{pmatrix}\mathbf{1}_d & w\\0 & \mathbf{1}_s\end{pmatrix}\begin{pmatrix}\xi_d & 0\\0 & \mathbf{1}_s\end{pmatrix} = \begin{pmatrix}\xi_d & w\\0 & \mathbf{1}_s\end{pmatrix}$$

$$\Rightarrow H_W = \xi_W^{-1}(\text{diag } H_W)\xi_W = \begin{pmatrix}H_d & \xi_d^{-1}(Mw - wm)\\0 & m\end{pmatrix}$$

$$\text{i.e., } D = \xi_d^{-1}(Mw - wm)$$

The decaying particles $|M\rangle$ have projections both on the orthogonal states $|U\rangle$ and on the stable particles $|m\rangle$

$$\text{particles: } \begin{pmatrix}|M\rangle\\|m\rangle\end{pmatrix} = \begin{pmatrix}\xi_d & w\\0 & \mathbf{1}_s\end{pmatrix}\begin{pmatrix}|U\rangle\\|m\rangle\end{pmatrix} = \begin{pmatrix}\xi_d|U\rangle + w|m\rangle\\|m\rangle\end{pmatrix}$$

The scalar product matrix for the decay collective with the $n = d + s$ particles arises from the diagonal matrix with the orthonormal states and the decay channels

$$\zeta_W = \xi_W\xi_W^* = \begin{pmatrix}\langle M \mid M\rangle & \langle M \mid m\rangle\\\langle m \mid M\rangle & \langle m \mid m\rangle\end{pmatrix} = \begin{pmatrix}\mathbf{1}_d & w\\0 & \mathbf{1}_s\end{pmatrix}\begin{pmatrix}\zeta_d & 0\\0 & \mathbf{1}_s\end{pmatrix}\begin{pmatrix}1_d & 0\\w^* & \mathbf{1}_s\end{pmatrix}$$

$$= \begin{pmatrix}\zeta_d + ww^* & w\\w^* & \mathbf{1}_s\end{pmatrix} \quad \text{with } \zeta_d = \xi_d\xi_d^*$$

The stable particles remain an orthogonal basis of the subspace $|m\rangle$.

The decomposition of the Hilbert space unit operator in the nonorthogonal particle basis displays the inverse scalar product matrix

$$\zeta_W^{-1} = \begin{pmatrix} \mathbf{1}_d & 0 \\ -w^* & \mathbf{1}_s \end{pmatrix} \begin{pmatrix} \zeta_d^{-1} & 0 \\ 0 & \mathbf{1}_s \end{pmatrix} \begin{pmatrix} \mathbf{1}_d & -w \\ 0 & \mathbf{1}_s \end{pmatrix} = \begin{pmatrix} \zeta_d^{-1} & -\zeta_d^{-1} w \\ -w^* \zeta_d^{-1} & \mathbf{1}_s + w^* \zeta_d^{-1} w \end{pmatrix}$$

$$\Rightarrow \mathbf{1}_n = |U\rangle\langle U| + |m\rangle\langle m|$$
$$= |M\rangle \zeta_d^{-1} \langle M| - |M\rangle \zeta_d^{-1} w \langle m| - |m\rangle w^* \zeta_d^{-1} \langle M| + |m\rangle(\mathbf{1}_s + w^* \zeta_d^{-1} w)\langle m|$$

To define probabilities and expectation values for unstable particles, a more general orientation with respect to the Hilbert space structures involved will be useful.

3. LOGIC OF QUANTUM THEORY (A SHORT REVIEW)

With Boole, Leibniz's dream of a formalization of logic which allows to draw conclusions in a mechanical way—like arithmetic computation—started to become realized. Apparently, logic condenses the structures of our experiences and, therewith, shows a close relationship to the formulations of physics. With the paramount importance of the complex linear superposition structure of quantum theory the classical Boolean logic, appropriate for classical phase space physics, gave way to a quantum logic as formulated by Birkhoff and von Neumann. As a consequence, the probability structure, already arising in classical physics, e.g. in thermostatistics, is not primary in quantum theory—it comes, so to say, as a square of a linear complex probability amplitude structure.

Nothing is new in the following section—it should serve as a short reminder and should introduce the concepts and notations used later on. In addition to Birkhoff-von Neumann's original article there is Varadarajan's detailed text-book which can be consulted for a deeper information.

3.1. Logics

The propositions of a logic are formalized as the elements of a lattice, i.e. of a set with two associative and commutative inner compositions (meet $\sqcap$ and join $\sqcup$) which have an absorptive relationship to each other

$$(L, \sqcap, \sqcup) \in \underline{\mathbf{latt}}: \; a \sqcup (a \sqcap b) = a = a \sqcap (a \sqcup b) \text{ (absorptive)}$$

Each lattice carries its natural order $a \sqsubseteq b \iff a \sqcap b = a$.

A lattice with an origin $\Box$—it is unique

$$\Box \sqsubseteq a, \quad \text{i.e. } \Box = \Box \sqcap a \qquad \text{for all } a \in L$$

allows the definition of disjoint elements by $a \sqcap b = \Box$.

A complementary lattice has an involutive contramorphism relating meet and join with the origin as meet for each lattice element and its complement

$$L \to L, \quad a \mapsto a^c, \quad a^{cc} = a \qquad \begin{cases} (a \sqcup b)^c = a^c \sqcap b^c \\ a \sqcap a^c = \square \quad \text{for all } a \in L \end{cases}$$

The complement of the origin is the unique end $\blacksquare$

$$\square^c = \blacksquare \sqsupseteq a, \quad a \sqcup a^c = \blacksquare \qquad \text{for all } a \in L$$

With an appropriate language for the logical concepts a complemented lattice is used as a logic

$$(L, \sqcap, \sqcup, \square, c) \in \underline{\textbf{logic}} : \begin{cases} a \in L: & \text{proposition} \\ \sqcap: & \text{conjunction (and, et)} \\ \sqcup: & \text{adjunction (or, aut)} \\ \sqsubseteq: & \text{implication (then, ergo)} \\ \square: & \text{absurd proposition (falsehood, falsum)} \\ a^c \in L: & \text{negation (not, non)} \\ \blacksquare: & \text{self-evident proposition (truth, verum)} \end{cases}$$

A lattice is distributive for

$$a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$$

$$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$$

Weaker than distributivity is modularity, a partial associativity for meet and join

$$a \sqsubseteq c \Rightarrow a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c$$

3.2. Boolean Logics

A distributive logic is, by Stone's theorem (Stone, 1936), isomorphic to a lattice of subsets $\mathcal{M} \subseteq 2^M = \{X \subseteq M\}$ with 2^M the power set of a set M. The lattice operations are intersection and union, the negation uses the set complement

$$\text{(distributive) } \underline{\textbf{logic}} \ni (L, \sqcap, \sqcup, \square, c) \cong (\mathcal{M}, \cap, \cup, \emptyset, C_M)$$

The valuation of a Boolean logic employs probability measures, i.e. disjoint additive mappings on the lattice with positive values between 0 for the falsum and 1 for the verum

$$\mu : \mathcal{M} \to \mathbb{R}_+ \quad \begin{cases} \mu(\emptyset) &= 0 \\ \mu(X \cup Y) &= \mu(X) + \mu(Y) \ \text{ for } X \cap Y = \emptyset \\ \mu(M) &= 1 \end{cases}$$

In the classical formulation of physics, the propositions in the corresponding Boolean logic are subsets of the phase space of a physical system. In deterministic classical mechanics, the measurements are formalized by the numerical values of phase space functions, i.e. the probabilities used are yes–no probabilities on point subsets $\{(x, p)\}$ of the phase space

$$X = \{(x, p)\} : \ \mu x : \{\emptyset, X\} \to \{0, 1\}, \quad \mu_X(X) = 1$$

In thermostatistics coarser subset lattices are used.

3.3. Birkhoff-von Neumann Logics

The subspaces $2^{\bar{V}} = \{W \subseteq V | \text{ closed subspace}\}$ of a Hilbert space V—in the following only over the abelian fields $\mathbb{K} = \mathbb{R}, \mathbb{C}$—with, again, the intersection as meet, but the span as join, the trivial space for the logical "falsum" and the orthocomplementation for the negation constitute a linear logic

$$(L, \sqcap, \sqcup, \square, c) \cong (2^{\bar{V}}, \cap, +, \{0\}, \perp) \in \underline{\mathbf{logic}} \text{ (linear)}$$

A Hilbert state space of quantum mechanics is used for a Birkhoff-von Neumann logic. It extends the set union for a Boolean logic by the quantum characteristic linear superposition. In the following, the relevant features of subspace lattices are reviewed.

For dimension $n \geq 2$ (where the vector space endomorphisms are nonabelian) linear lattices are not distributive (basis $\{e^i\}$)

$$W_i = \mathbb{K}e^i \cong \mathbb{K}, \quad i = 1, 2 \qquad \begin{array}{l} \text{full space: } V = W_1 + W_2 \cong \mathbb{K}^2 \\ \text{diagonal space: } \Delta = \mathbb{K}(e^1 + e^2) \cong \mathbb{K} \end{array}$$

$$(W_1 + W_2) \cap \Delta = \Delta \neq (W_1 \cap \Delta) + (W_2 \cap \Delta) = \{0\} + \{0\} = \{0\} = W_1 \cap W_2$$

A lattice with finite dimensional subspaces is modular.

Linear lattices can be "operationalized," i.e. they can be embedded into the endomorphisms of the full vector space V, denoted by $\mathbf{AL}(V)$—a unital $\mathbb{K}$-algebra, for finite dimensions $\mathbf{AL}(V) \cong V \otimes V^T$ with the dual space V^T. Any idempotent $\mathcal{P}$ (projector for $\mathcal{P} \neq 0$) defines a subspace W and—by its kernel—a direct complement W'

$$\begin{aligned} \mathbf{AL}(V) \ni \mathcal{P} = \mathcal{P}^2 \mapsto W = \mathcal{P}(V) \in 2^V \\ V = W \oplus W' \quad \text{with } W' = \mathcal{P}^{-1}(0) \in 2^V \end{aligned}$$

One subspace can be defined by different projectors and can have different complements—in the example above with two different dual bases

$$\begin{aligned} \mathrm{id}_V &= \mathcal{P}_1 + \mathcal{P}_2 = e^1 \otimes \check{e}_1 + e^2 \otimes \check{e}_2 \\ &= \mathcal{P}_1' + \mathcal{P}_\Delta = e^1 \otimes (\check{e}_1 - \check{e}_2) + (e^1 + e^2) \otimes \check{e}_2 \end{aligned}$$

$$W_1 = \mathcal{P}_1(V) = \mathcal{P}'_1(V), \qquad V = W_1 \oplus W_2 = W_1 \oplus \Delta$$

Uniqueness can be obtained with a dual isomorphism $V \overset{\zeta}{\leftrightarrow} V^T$.

With a nondegenerate inner product (symmetric bilinear or sesquilinear form)

$$\langle|\rangle : V \times V \to \mathbb{K}, \zeta(v, w) = \langle v \mid w\rangle = \overline{\langle v|w\rangle}$$
$$\langle v \mid w + u\rangle = \langle v \mid w\rangle + \langle v \mid u\rangle, \langle v \mid \alpha w\rangle = \alpha\langle v \mid w\rangle$$
$$\langle v \mid V\rangle = \{0\} \Leftrightarrow v = 0$$

each subspace has a unique orthogonal subspace partner

$$\perp: 2^V \to 2^V, \quad W \mapsto W^\perp = \{v \in V | \langle W \mid v\rangle = \{0\}\}$$

In a finite dimensional space V orthogonality defines an involution

$$V \cong \mathbb{K}^n : W = W^{\perp\perp}$$

With a nondegenerate square, projectors are bijectiveley related to subspaces

$$2^V \ni W \overset{\zeta}{\leftrightarrow} \mathcal{P}_W \in \mathbf{AL}(V), \quad W = \mathcal{P}_W(V)$$

i.e., the lattice of vector subspaces can be considered to be operators.

The dual isomorphism allows the bra–ket notation (next subsection) wherewith the projector for a finite dimensional subspace $W \cong \mathbb{K}^k$ can be written with a W-basis $\{e^\kappa\}_{\kappa=1}^k$ (summation convention)

$$\mathcal{P}_W = |e^\kappa\rangle\zeta_{\lambda\kappa}\langle e^\lambda| \quad \text{with } \langle e^\lambda \mid e^\mu\rangle = \zeta^{\mu\lambda}, \qquad \zeta^{\mu\lambda}\zeta_{\lambda\kappa} = \delta_\kappa^\mu$$
$$\mathrm{tr}_V\, \mathcal{P}_W = \zeta_{\lambda\kappa}\zeta^{\kappa\lambda} = \delta_\kappa^\kappa = d(W)$$

The involution defined by orthogonality is not complementary for an indefinite nondegenerate square. E.g., in a 2-dimensional $\mathbf{O}(1,1)$ Minkowski space with Lorentz metric $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ in a basis $\{e^0, e^3\}$, time and position translations $\mathbb{T}$ and $\mathbb{S}$ resp. are orthogonal to each other whereas the isotropic lightlike subspaces $\mathbb{L}_\pm$ are self-orthogonal

$$\mathbb{T}^\perp = (\mathbb{R}e^0)^\perp = \mathbb{R}e^3 = \mathbb{S}$$
$$\alpha\beta \neq 0 \Rightarrow \mathbb{R}(\alpha e^0 + \beta e^3)^\perp = \mathbb{R}\left(\frac{1}{\alpha}e^0 + \frac{1}{\beta}e^3\right) \Rightarrow \mathbb{L}_\pm = \mathbb{R}(e^0 \pm e^3) = \mathbb{L}_\pm^\perp$$
$$\mathcal{P}_\mathrm{T} = |e^0\rangle\langle e^0|, \quad \mathcal{P}_\mathrm{S} = -|e^3\rangle\langle e^3|, \quad \mathcal{P}_{\mathrm{L}\pm} = \frac{1}{2}|e^0 \pm e^3\rangle\langle e^0 \mp e^3|$$

A complementary linear lattice has to come with a definite square, i.e. with a scalar product, which is nondegenerate in each subspace

$$\langle v \mid v \rangle = 0 \Longleftrightarrow v = 0 \Rightarrow \mathbb{K}^n \cong V = W + W^\perp = W \oplus W^\perp = W \perp W^\perp$$

An inner product of $V \cong \mathbb{C}^n$ is positive if, and only if, it is a product $\zeta_n = SS^*$ of an endomorphism $S \in \mathbf{AL}(V)$ and its $\mathbf{U}(n)$-hermitian S^*.

Now the probability valuation of vector space sublattices: With a scalar product ζ, each nontrivial finite dimensional vector space W carries a yes–no probability with a normalized discriminant

$$\zeta \succeq 0 : \mu_W : \{\emptyset, W\} \to \{0, 1\}$$
$$\mu_W(W) = \det \zeta_W = \det \langle e^\lambda \mid e^\mu \rangle = 1$$

The classical measure comes as the positive scalar product.

The Schrödinger wave functions—possible for Bose structures, e.g. the harmonic Bose oscillator above $|k\rangle \cong \psi_k(x)$—as position translation orbits allow a "smearing out" of the probabilities for the 1-dimensional subspaces (Hilbert rays $W = \mathbb{C}|k\rangle$)

$$\det \zeta_W = \langle k \mid k \rangle = \int_{\mathrm{R}} dx |\psi_k(x)|^2 = 1$$

to position densities for the probability, here $|\psi_k(x)|^2$.

For finite dimension $V \cong \mathbb{K}^n$, the trace is an invariant linear form with the trace of a projector giving the dimension of the defined subspace

$$\mathrm{tr}_V : \mathbf{AL}(V) \to \mathbb{K}, \qquad f \mapsto \mathrm{tr}_V\ f$$
$$\mathrm{tr}_V\ \mathcal{P} = \dim_K \mathcal{P}(V)$$

The expectation values $\mathcal{E}_W$ in the subspace W of the operating algebra elements uses the trace, normalized with the dimension $\mathrm{tr}\ \mathcal{P}_W = d(W)$

$$\mathbf{AL}(V) \ni f \mapsto \mathcal{E}_W(f) = \frac{1}{d(w)} \mathrm{tr}_V\ \mathcal{P}_W \circ f = \frac{\zeta_{\lambda\kappa} \langle e^\lambda | f | e^\kappa \rangle}{d(W)} \in \mathbb{K}$$

The expectation values in W of the other subspaces use their operational form as projectors

$$\mathcal{E}_W : 2^V \to \mathbb{R}_+ \quad \begin{cases} \mathcal{E}_W(U) = \mathcal{E}_W(\mathcal{P}_U) = \frac{1}{d(W)} \mathrm{tr}_V\ \mathcal{P}_W \circ \mathcal{P}_U \in [0, d(U)] \\ \mathcal{E}_W(U_1 \perp U_2) = \mathcal{E}_W(U_1) + \mathcal{E}_W(U_2) \\ U \subseteq W \Rightarrow \mathcal{E}_W(U) = \frac{d(U)}{d(W)} \\ \mathcal{E}_W(V) = \mathcal{E}_W(\mathrm{id}_V) = \frac{1}{d(W)} \mathrm{tr}_V\ \mathcal{P}_W = 1 = \mathcal{E}_W(W) \end{cases}$$

A familiar special case are the symmetric transition probabilities between two 1-dimensional spaces ($i = 1, 2$)

$$W_i = \mathbb{K}|e^i\rangle, \quad \mathcal{P}_{W_i} = |e^i\rangle\langle e^i| \Rightarrow \mathcal{E}_{W1}(W_2) = \mathcal{E}_{W_2}(W_1) = |\langle e^1 \mid e^2 \rangle|^2 \in [0, 1]$$

A basis formulation reads as follows—each subspace comes with a basis (different indices)

$$\left.\begin{array}{l}\mathcal{P}_W = |e^\kappa\rangle\zeta_{\lambda\kappa}\langle e^\lambda| \\ \mathcal{P}_U = |e^A\rangle\zeta_{BA}\langle e^B|\end{array}\right\} \Rightarrow \left\{\begin{array}{r}\mathcal{P}_W \circ \mathcal{P}_U = |e^\kappa\rangle\zeta_{\lambda\kappa}\langle e^\lambda \mid e^A\rangle\zeta_{BA}\langle e^B| \\ \mathrm{tr}_V\ \mathcal{P}_W \circ \mathcal{P}_U = \zeta_{\lambda\kappa}\zeta_{BA}\langle e^\lambda \mid e^A\rangle\overline{\langle e^\kappa \mid e^B\rangle}\end{array}\right.$$

especially simple for Euclidean bases

$$\zeta_{\lambda\kappa} = \delta_{\lambda\kappa}, \quad \zeta_{BA} = \delta_{BA} \Rightarrow \mathrm{tr}_V\ \mathcal{P}_W \circ \mathcal{P}_U = \sum_{\lambda,A} |\langle e^\lambda \mid e^A\rangle|^2 \leq d(W)d(U)$$

3.4. Hilbert-Beins for Particle Collectives

For a basis $\{e^\kappa|\kappa = 1, \ldots, n\}$ of the Hilbert space $W \cong \mathbb{C}^n$ the scalar product gives the matrix

$$\zeta : W \times W \to \mathbb{C}, \quad \langle e^\lambda \mid e^\kappa\rangle = \zeta^{\kappa\lambda} = \overline{\zeta^{\lambda\kappa}}$$

with the inverse scalar product on the dual space (linear forms) with the dual basis $\{e_\kappa|\kappa = 1, \ldots, n\}$

$$\zeta^{-1} : W^T \times W^T \to \mathbb{C}, \quad \langle e_\lambda \mid e_\kappa\rangle = \zeta_{\kappa\lambda} = \overline{\zeta_{\lambda\kappa}}$$

The dual isomorphism $W \cong W^T$, induced by a nondegenerate product, allows Dirac's bra-ket notation

$$\left.\begin{array}{l} e^\kappa = |e^\kappa\rangle \\ e_\kappa = \langle e^\lambda|\zeta_{\lambda\kappa}\end{array}\right\} \Rightarrow \text{dual product: } \delta^\kappa_\mu = \langle e_\mu, e^\kappa\rangle = \langle e^\lambda \mid e^\kappa\rangle\zeta_{\lambda\mu} = \zeta^{\kappa\lambda}\zeta_{\lambda\mu}$$

Using the dual isomorphism, which is antilinear for a sesquilinear form ζ, a linear transformation of W can be expressed in the bra-ket notation

$$\begin{aligned} f : W \to W, \quad f = f^\lambda_\kappa e^\kappa \otimes e_\lambda &= |e^\kappa\rangle\zeta_{\lambda\mu} f^\mu_\kappa\langle e^\lambda| \\ \langle e^\lambda| f |e^\kappa\rangle &= f^\kappa_\mu\zeta^{\mu\lambda} = f^{\kappa\lambda}\end{aligned}$$

Orthonormal bases $\{e^a|a = 1, \ldots, n\}$ are related to the basis $\{e^\kappa\}$ by a W-automorphism ξ (n-bein in the Hilbert space)

$$\langle e^b \mid e^a\rangle = \delta^{ab}, \quad \begin{cases}\xi : W \to W, & e^\kappa = |e^\kappa\rangle \mapsto \xi^\kappa_a|e^a\rangle \\ \xi^{-1T} : W^T \to W^T, & e_\kappa \mapsto (\xi^{-1})^a_\kappa e_a \\ & e_\kappa\zeta^{\kappa\lambda} = \langle e^\lambda| \mapsto (\xi^*)^\lambda_a\langle e^a|\end{cases}$$

Bases of translation eigenstates describing unstable particles have not to be orthonormal. An unstable particle collective $W \leftrightarrow \mathcal{P}_W$ comes with its Hilbert-bein ξ_W.

The state space metric is the absolute square of the Hilbert-bein

$$\begin{aligned}&\langle e^\lambda \mid e^\kappa\rangle = \zeta^{\kappa\lambda} = (\xi^*)^\lambda_b\delta^{ab}\xi^\kappa_a, \quad \zeta = \xi\delta\xi^* \\ &\langle e_\lambda \mid e_\kappa\rangle = \zeta_{\kappa\lambda} = (\xi^{-1})^b_\lambda\delta_{ab}(\xi^{-1*})^a_\kappa, \quad \zeta^{-1} = \hat{\xi}\delta\hat{\xi}^*, \ \hat{\xi} = \xi^{-1*}\end{aligned}$$

The positivity of the Hilbert product is seen in the matrix product form—any product ff^* of $(n \times n)$-matrices is positive, i.e. has positive spectrum.

The Hilbert-bein ζ arises in inner automorphisms for linear transformations—in the bra-ket formulation

$$\langle e^\lambda | f | e^\kappa \rangle = (\xi^*)^\lambda_b \langle e^b | f | e^a \rangle \xi^\kappa_a$$

Obviously, the structures above with the bra-ket formalism and the transition from orthonormal to general bases constitute the sesquilinear product analogue (Saller, 1998) of the more familiar structures with a bilinear real metric g, e.g. for real 4-dimensional spacetime the raising and lowering of indices with g and g^{-1}. The metric is the square of the diagonalizing tetrad (vierbein) h

$$g^{\mu\nu} = h^\nu_j \eta^{kj} h^\mu_k, \quad g = h\eta h^T$$

with flat Minkowski space orthogonal metrical matrix $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix}$. The transposition T in the real bilinear case is replaced by the conjugate transposition $*$ for the complex Hilbert space. The spacetime metric discriminant $\det g = -(\det h)^2$ has its analogue in the discriminant of the Hilbert product

$$\det \zeta = |\det \xi|^2$$

which is used for the probability normalization of particle transition amplitudes.

What is not analogue for spacetime metric and vierbein, on the one side, and Hilbert space product and Hilbert-bein, on the other side, is the real 4-dimensional spacetime dependence of the tetrad (metric) $x \mapsto h(x)$ which has no counterpart in the Hilbert-bein and state space metric. In addition there is the important difference that the bilinear metrical matrix represents a tensor $g(x) \in \mathbb{M}(x) \otimes \mathbb{M}(x)$ of the tangent Minkowski translations $\mathbb{M}(x) \cong \mathbb{R}^4$ whereas the sesquilinear scalar product matrix of the Hilbert space is no tensor $\zeta \notin W \otimes W$. Raising and lowering indices with the Hilbert space metric ζ, e.g. in $\langle e^\lambda | f | e^\kappa \rangle = f^\kappa_\mu \zeta^{\mu\lambda}$, changes bilinearity into sesquilinearity.

A spacetime tetrad represents (at each spacetime point x) a class of the real 10-dimensional symmetric space with the Lorentz groups in the general linear group

$$\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1,3) \cong \mathbf{D}(1) \times \mathbf{SL}_0(\mathbb{R}^4)/\mathbf{SO}_0(1,3)$$

with the "overall" dilatation group $\mathbf{D}(1) = \exp \mathbb{R}$ and the orthochronous Lorentz group $\mathbf{SO}_0(1, 3)$. The classes are characterized by the value of the similarity invariants which can be found in the coefficients of the characteristic polynomial $\det(\log h - X\mathbf{1}_4)$ for the tetrad generator. The tetrad manifold has two continuous invariants $\{\Delta_0, \Delta\}$ which can be obtained also by diagonalization of the symmetric metrical matrix $g = g^T$ with an orthogonal transformation O and and a double hyperbolic dilatation transformation D, the latter one equalizing the dilatations for the three space directions

$$g = OD\,\text{diag}\,gD^T O^T \quad \text{with} \quad O \in \mathbf{SO}(4),\ D \in \mathbf{SO}_0(1,1)^2$$
$$\text{diag}\,g = \begin{pmatrix} e^{2\Delta_0} & 0 \\ 0 & -\mathbf{1}_3 e^{2\Delta} \end{pmatrix} = \frac{\ell^2}{\sqrt{c}} \begin{pmatrix} \frac{1}{\sqrt{c^3}} & 0 \\ 0 & -\mathbf{1}_3\sqrt{c} \end{pmatrix}$$

The two continuous invariants for the local rescaling of time and position (local time and local length unit)

$$(dx_0, d\vec{x}) \mapsto \left(e^{\Delta_0(x)}dx_0, e^{\Delta(x)}d\vec{x}\right), \quad e^{\Delta_0(x)} = \frac{\ell(x)}{c(x)}, \quad e^{\Delta(x)} = \ell(x)$$

arise from the invariant $\det h = \frac{\ell^4}{c}$ for the overall dilatation $\mathbf{D}(1)$ and the invariant c (local velocity unit) for the subgroup $\mathbf{SO}_0(1,1)$ in $\mathbf{SL}_0(\mathbb{R}^4)/\mathbf{SO}_0(1,3)$.

As for the Hilbert space metric, a Hilbert-bein represents a class of the real n^2-dimensional manifold with the unitary groups in the general linear group

$$\mathbf{GL}(\mathbb{C}^n)/\mathbf{U}(n) \cong \mathbf{D}(1) \times \mathbf{SL}(\mathbb{C}^n)/\mathbf{SU}(n)$$

with one invariant for $\mathbf{D}(1)$ and $n-1$ invariants for the special factor. All n-invariants (similarity invariants of the Hilbert-bein generator in $\det[\log\xi - X\mathbf{1}_n]$) are taken from a continuous spectrum and can be found with the manifold isomorphy

$$\mathbf{SU}(n) \cong \mathbf{SO}(2)^{n-1} \times \mathbf{SU}(n)/\mathbf{SO}(2)^{n-1}$$
$$\mathbf{SL}(\mathbb{C}^n)/\mathbf{SU}(n) \cong \mathbf{SO}_0(1,1)^{n-1} \times \mathbf{SU}(n)/\mathbf{SO}(2)^{n-1}$$

by a special unitary diagonalization of the hermitian scalar product matrix

$$\zeta = \zeta^*$$
$$\zeta = \xi\delta\xi^* = U\,\text{diag}\,\zeta U^* \quad \text{with } U \in \mathbf{SU}(n)$$
$$\text{diag}\,\zeta = e^{2\Delta_0} \begin{pmatrix} e^{2\Delta_1} & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & e^{2\Delta_n} \end{pmatrix} \in \mathbf{D}(1) \times \mathbf{SO}_0(1,1)^{n-1}, \qquad \sum_{k=1}^{n} \Delta_k = 0$$

The non-orthogonality of unstable particles gives rise to invariants $\{\Delta_k\}_{k=1}^n$ in a normalized Hilbert-bein ξ, $\det\xi = 1$, which are related to the characteristic decay parameters.

4. UNSTABLE STATES AND PARTICLES (PART 2)

4.1. The Unitarity of Particles

For stable particles with mass m, Wigner's definition (Wigner, 1939) is used, characterizing a particle as a vector acted upon with a unitary irreducible representation of the Poincaré group which—because of the noncompact nonabelian degrees of freedom—has to be infinite dimensional as expressed by the integral above $\int_{\mathbf{R}^3} \frac{d^3p}{(2\pi)^3} \frac{}{(2\pi)^3 p_0}$ where $\mathbb{R}^3$ for the momenta parametrizes the boost cosets $\mathbf{SO}_0(1,3)/\mathbf{SO}(3)$. With Wigner's definition, confined quarks are no particles, they are no eigenvectors with respect to the spacetime translations, i.e. they have no invariant translation eigenvalue (particle mass).

For masses $m^2 \geq 0$, the representations of the Poincaré covering group $\mathbf{SL}(\mathbb{C}^2) \vec{\times} \mathbb{R}^4$ with the orthochronous Lorentz covering group $\mathbf{SL}(\mathbb{C}^2)/\{\pm \mathbf{1}_2\} \cong \mathbf{SO}_0(1,3)$ are induced from the unitary representations of the direct product little groups with the spin or polarization group for the space rotations

$$\text{for } m^2 > 0 : \mathbf{SU}(2) \times \mathbb{R} \to \mathbf{U}(1+2J)$$
$$\text{for } m^2 = 0 : \mathbf{SO}(2) \times \mathbb{R} \to \mathbf{U}(1+2|J_3|)$$

The translations in the direct product groups can be taken to be time translations, e.g. in a rest system for $m^2 > 0$. They come in harmonic oscillator representations as given in the 1st section and represented in the phase group, i.e. in $\mathbf{U}(1) \cong \mathbf{U}(1+2J)/\mathbf{SU}(1+2J)$

$$\mathbb{R} \ni t \mapsto e^{iEt} \in \mathbf{U}(1) \quad \text{with } E \in \mathbb{R}$$

In contrast to the compact position rotations the noncompact translations have also representations in noncompact groups. Unstable particles with a nontrivial positive width are orbits of not unitary irreducible time representations

$$\mathbb{R}_+ \ni t \mapsto D(t) = e^{(iE-\frac{\Gamma}{2})t} \notin \mathbf{U}(1) \quad \text{with } \Gamma > 0$$

They can be used only for the future monoid $\mathbb{R}_+$. Unitarity as necessary for a complex representation of a real group is restored by taking the direct sum with the anti-representation (inverse-conjugated) for the past monoid $\mathbb{R}_-$

$$\mathbb{R}_- \ni t \mapsto D(-t)^* = e^{\left(iE+\frac{\Gamma}{2}\right)t} \notin \mathbf{U}(1)$$

The resulting representation is indefinite unitary (Saller, 2001)

$$\mathbb{R} \ni t \mapsto D(t) \oplus D(-t)^* = e^{iEt} \begin{pmatrix} e^{-\frac{\Gamma}{2}t} & 0 \\ 0 & e^{+\frac{\Gamma}{2}t} \end{pmatrix} \in \mathbf{U}(1,1) \subset \mathbf{GL}(\mathbb{C}^2)$$

leaving invariant the indefinite square in the off-diagonal isotropic basis

$$\zeta_{(1,1)} : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}, \qquad \zeta_{(1,1)} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which cannot be used for a Hilbert space product. It is possible to use a 2nd decomposable conjugation $\mathbf{U}(1) \times \mathbf{U}(1) \subset \mathbf{U}(2)$ with scalar product $\zeta_2 \cong \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in the 2-dimensional space. Then unstable particles are described by non-unitary future monoid representations in a Hilbert space, i.e.—in contrast to stable particles—the scalar product invariance group defining the probability does not contain the represented time translations. Probabilities have a nontrivial time dependence.

Such a $\mathbf{U}(1, 1)$-representation of the translations can be used—as for stable particles—to define corresponding representations of the Poincaré monoid $\mathbf{SL}(\mathbb{C}^2) \vec{\times} \mathbb{R}^4_+$ with $\mathbb{R}^4_+$ the future spacetime translation cone. In addition to the energy width there arises also a momentum spread and, therewith, nontrivial spin mixtures for unstable particles (Blum and Saller, in preparation).

These were some short remarks to bypass a not satifactorily solved problem—how to reconcile the different unitarities for rotations (definite) and spacetime translations (possibly indefinite) with the probability interpretation (necessarily definite).

4.2. Nondecomposable Particle Collectives

In the “huge” Hilbert space with all particles there are—neglecting the momentum dependence $\vec{p} \in \mathbb{R}^3$—1-dimensional subspaces connected with stable particles and higher dimensional ones for decay collectives. With respect to probabilities and expectation values, those subspaces have to be considered as a “whole.” This can be seen in some analogy to a relativistic spacetime vector with many bases for time and space projections $x = (x_0, \vec{x})$, but with only one Lorentz length x^2.

It is assumed that the “huge” Hilbert space has a basis with particle states (translation eigenstates), stable and unstable. It is decomposable into nondecomposable orthogonal subspaces $\langle W|U\rangle = \{0\}$ for $W \neq U$, assumed to be finite dimensional (always neglecting the continuous momentum dependence $\vec{p} \in \mathbb{R}^3$).

A basis with n particle states in an orthogonally nondecomposable subspace $W \cong \mathbb{C}^n$ with a corresponding positive scalar product matrix $\zeta_W = \xi_W \xi_W^*$ and Hilbert-bein ξ_W will be probability-normalized by its discriminant, $\det \zeta_W = 1$, and with trivial phase, i.e. the Hilbert-bein ζ_W involves maximally $(n^2 - 1)$-real parameters of a noncompact class $\mathbf{SL}(\mathbb{C}^n)/\mathbf{SU}(n)$ with $(n-1)$ continuous invariants

$$\text{scalar product } \zeta_W = \xi_W \xi_W^* \succeq 0 \quad \text{with} \quad \det \xi_W = 1$$

A nondecomposable 1-dimensional space is the ray of a stable particle state

$$W = \mathbb{C}|m\rangle \quad \text{with} \quad \det \zeta_1 = \langle m \mid m\rangle = 1$$

An orthogonally nondecomposable space with $n \geq 2$ describes a decay collective $W = |M\rangle \oplus |m\rangle \cong \mathbb{C}^n$ as discussed above: The Hilbert-bein as transformation from orthogonal to particle basis can be brought to a triangular form $\xi_W = \left(\begin{smallmatrix} \xi_d & w \\ 0 & \mathbf{1}_s \end{smallmatrix}\right)$, $w \neq 0$, with a unit submatrix leaving invariant the subspace $|m\rangle \cong \mathbb{C}^s$ with $s \geq 1$ stable states and a non-orthogonal complement $|M\rangle \cong \mathbb{C}^d$ with $d \geq 1$ unstable states. The discriminant normalization of the full scalar product matrix coincides with the discriminant normalization for the unstable states

$$\begin{aligned} \det \zeta_W &= \langle \det \xi_W \mid \det \xi_W \rangle = \langle M \mid M\rangle\langle m \mid m\rangle - \langle M \mid m\rangle\langle m \mid M\rangle \\ &= \det \begin{pmatrix} \zeta_d + ww^* & w \\ w^* & \mathbf{1}_s \end{pmatrix} = \det \zeta_d = \langle \det \xi_d \mid \det \xi_d \rangle = 1 \end{aligned}$$

The projector for the decay collective involves the inverse scalar product matrix ζ_W^{-1}

$$\mathcal{P}_W = |M\rangle\zeta_d^{-1}\langle M| - |M\rangle\zeta_d^{-1}w\langle m| - |m\rangle w^*\zeta_d^{-1}\langle M| + |m\rangle(\mathbf{1}_s + w^*\zeta_d^{-1}w)\langle m|$$

The projectors for the non-orthogonal subspaces are

$$\mathcal{P}_{|M\rangle} = |M\rangle(\zeta_d + ww^*)^{-1}\langle M|, \qquad \mathcal{P}_{|m\rangle} = |m\rangle\mathbf{1}_s\langle m|$$

The probability normalization for the kaon system with the discriminant is collective: It involves the decay parameter, i.e. the non-orthogonality δ with $0 < \delta^2 < 1$

$$\begin{aligned} \det \zeta_2 &= \langle M_S \mid M_S\rangle\langle M_L \mid M_L\rangle - \langle M_S \mid M_L\rangle\langle M_L \mid M_S\rangle \\ &= \det \frac{1}{|N|^2}\begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix} = \frac{1-\delta^2}{|N|^4} = 1 \Rightarrow |N|^2 = \sqrt{1-\delta^2} \end{aligned}$$

and differs from the individual probability normalization for each particle which would be given by $|N|^2 = 1$. The continuous invariant Δ related to the rank 1 of the Hilbert-bein manifold $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SO}_0(1,1) \times \mathbf{SU}(2)/\mathbf{SO}(2)$ is seen in the $\mathbf{SO}_0(1,1)$-adapted parametrization of the kaonscalar product

$$\begin{pmatrix} \langle M_S \mid M_S\rangle & \langle M_S \mid M_L\rangle \\ \langle M_L \mid M_S\rangle & \langle M_L \mid M_L\rangle \end{pmatrix} = \begin{pmatrix} \cosh\Delta & \sinh\Delta \\ \sinh\Delta & \cosh\Delta \end{pmatrix} \sim \begin{pmatrix} e^{\Delta} & 0 \\ 0 & e^{-\Delta} \end{pmatrix}, \quad \tanh\Delta = \delta$$

Neglecting weak and electromagnetic interactions, the neutron and pion, e.g., are stable. Taking into account the mentioned interactions, each of these particles constitutes the decaying subspace of a decay collective, e.g. for the neutron $\{|n\rangle, |p, e, \overline{\nu}_e\rangle\}$ with a nontrivial projection $\langle n|p, e, \overline{\nu}_e$. "Switching on" all interactions there seems to exist only a small number of high dimensional orthogonally

Invariants for Nondecomposable Collectives

Particle (lowest mass representative	Mass for translations $\mathbb{R}^4$ m^2	Spin **SU**(2) or polarization **SO**(2) J or $\pm\|J_3\|$	Charge **U**(1) Q	Fermion number **U**(1) F
Photon	0	± 1	0	0
(Anti) proton	>0	$\frac{1}{2}$	± 1	± 1
(Anti)electron	>0	$\frac{1}{2}$	∓ 1	± 1
(Anti)neutrino	>0 or $=0$ (?)	$\frac{1}{2}$ or $\pm\frac{1}{2}$ (?)	0	± 1

nondecomposable particle collectives which span the particle Hilbert space. Their particle representatives with lowest mass (the lowest step in the staircase) reflect the few invariants of relativistic particle physics, i.e. the mass for spacetime translations $\mathbb{R}^4$, the rotation invariants, characterizing spin **SU**(2) for massive particles and polarization **SO**(2) for massless ones and the electromagnetic charge number for a phase group **U**(1). If the proton is stable, there has to be an additional invariant, usually related to fermion number F conservation which is taken care of with the different association of charge and fermion number for proton with $Q + F = 2$ and electron with $Q + F = 0$. In addition, there may exist invariants for the leptonic phases—electronic, muonic, and tauonic.

Stable and unstable particle states come on the same level as Hilbert space directions—stable particles, e.g. the electron, are not “more fundamental” than unstable ones, e.g. the muon or the pion. An S-matrix with only stable in- and out-particle state vectors where the unstable ones are taken care of as intermediate fictive poles only (Weinberg, 1995) is against a democratic treatment. Depending on the degree of approximation to the distinction of stable–unstable as quantified in the magnitude of the off-diagonal entries w in the Hilbert-bein ξ_W, one may work with a larger or a smaller number of nondecomposable collectives, i.e. with a smaller or a larger number of stable particles.

4.3. (Non) Unitary *S*-Matrix for Unstable Particles

As an example, how the collective higher dimensional structure affects the probability interpretation, the unitarity structure of the S-matrix, involving a scattering with unstable particles, is considered.

Starting from a free Hamiltonian H_o acting on a Hilbert space with an eigenvector basis

$$H_0|E\rangle = E|E\rangle$$

the in and out states for an interaction Hamiltonian H are assumed to be constructable by inner automorphisms with the Moeller operators $\Omega_\pm$ for infinite future and past (Weinberg, 1995)

$$H = \Omega_\pm H_0 \Omega_\pm^{-1} \Rightarrow H|E_\pm\rangle = E|E_\pm\rangle \text{ for } |E_\pm\rangle = \Omega_\pm |E_\pm\rangle$$

The scattering operator is the product of the Moeller operators

$$S = \Omega_+^* \Omega_-$$

In quantum mechanics the Moeller operators are assumed to arise as limits

$$\Omega(t) = e^{iHt} e^{-iH_0 t}, \qquad \Omega_\pm = \lim_{t\to\pm\infty} \Omega(t),$$

Unitarity for hermitian Hamiltonians is assumed to survive the limit

$$\text{if } H_0 = H_0^*, \quad H = H^* \Rightarrow \Omega(t)^* = \Omega(t)^{-1}, \quad \Omega_\pm^* = \Omega_\pm^{-1} \Rightarrow S^* = S^{-1}$$

The scattering amplitudes are not matrix elements of linear transformations, but scalar products of in and out states—sesquilinear, not bilinear structures. They start with the scalar product matrix of the free particles

$$\left\langle E_+^\lambda \mid E_-^\kappa \right\rangle = \langle E^\lambda|S|E^\kappa\rangle = \langle E^\lambda|E^\kappa\rangle - 2i\pi \langle E^\lambda|T|E^\kappa\rangle, \qquad S = \mathbf{1} - 2i\pi T$$

If a decay collective is involved, the S-matrix $S^{\kappa\lambda}$ is not unitary

$$S^{\kappa\lambda} = \langle E^\lambda|S|E^\kappa\rangle = \zeta^{\kappa\lambda} + \cdots = \xi_b^\lambda \delta^{ba} (\xi^*)_a^\kappa + \cdots$$

Unitarity is expected for the S-matrix S_c^a transformed with the Hilbert-bein from a nonorthonormal particle basis into an orthonormal nonparticle basis

$$\hat{\xi}_\lambda^b \langle E^\lambda|S|E^\kappa\rangle (\hat{\xi}^*)_\kappa^a = \langle b|S|a\rangle = S_c^a \delta^{cb}$$

ACKNOWLEDGMENT

I am indebted to Walter Blum for discussions on the “identity overlap” for decay collectives as seen in the nontrivial scalar products involving unstable particles and their decay products.

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